String Wave Solutions

A solution to the wave equation for an ideal string can take the form of a traveling wave

$$y(x,t) = A \sin \frac{2\pi}{\lambda}(x - vt) \quad \text{or} \quad y(x,t) = A \sin \frac{2\pi}{\lambda}(x + vt)$$ Show

For a string of length L which is fixed at both ends, the solution can take the form of standing waves:

$$y(x,t) = A \sin \omega_n t \sin \frac{n\pi x}{L}$$ Show

For different initial conditions on such a string, the standing wave solution can be expressed to an arbitrary degree of precision by a Fourier series

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos \omega_n t \sin \frac{n\pi x}{L}$$

Calculation of parameters	Wave packet solution

String Standing Waves

For an ideal string of length L which is fixed at both ends, the solutions to the wave equation can take the form of standing waves:

$$y(x,t) = A \sin \omega_n t \sin \frac{n\pi x}{L}$$

This kind of solution can be verified by direct substitution into the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

Substituting:

$$\frac{\partial^2 y}{\partial x^2} = \frac{n^2\pi^2}{L^2} \sin \omega_n t \sin \frac{n\pi x}{L} \qquad \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\rho}{T}\omega_n^2 \sin \omega_n t \sin \frac{n\pi x}{L}$$

These two expressions are equal for all values of x and t provided

$$\omega_n = \sqrt{\frac{T}{\rho}}\frac{n\pi}{L}$$

Since the wave speed is $v = \sqrt{\frac{T}{\rho}}$, the frequency condition can be expressed as

$$f_n = \frac{nv}{2L} \quad \text{where } n = 1, 2, 3, \ldots$$